Influence of expansion on hierarchical structure

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We study a one-dimensional model of gravitational instability in an Einstein–de Sitter universe. Scaling in both space and time results in an autonomous set of coupled Poisson-Vlasov equations for both the field and phase space density, and the *N*-body problem. Using dynamical simulation, we find direct evidence of hierarchical clustering. A multifractal analysis reveals a bifractal geometry similar to that observed in the distribution of galaxies. To demonstrate the role of scaling, we compare the system to other one-dimensional models recently employed to study structure formation. Finally we show that the model yields an estimate of the time of galaxy formation of the correct order.

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I. INTRODUCTION

The discovery nearly 25 years ago of large scale structures in the universe [1] stimulated several new approaches to cosmology. In order to explain, or at least model, the hierarchical distribution of galaxies in clusters, and clusters of clusters, surrounding immense voids, it was proposed that matter in the universe occupies a fractal set, with a definite fractal geometry and dimension [1,2]. It is currently believed that the process for creating this singular distribution of mass follows from the hydrodynamic flow of dark matter, either hot or cold [1,3]. In the former case structure evolves in the top down mode, while the latter is bottom up and reflects the more prevalent view. Both of these themes introduce their own peculiar set of difficulties: the fractal assumption plays havoc with the homogeneity at large scales required by the standard model [1] while the bottom up scenario may be problematic because it predicts galaxy formation at a time when the background density is small [4].

Before (and after) the discovery of large scales, useful information was obtained by assembling statistical information concerning the distribution of galaxies in the sky. The primary tool for organizing this information is the construction of correlation functions representing the distribution of pairs of galaxies, triples, etc. While complete functions at all orders are unobtainable, specific symmetries and scaling laws have emerged from the data. First the validity of a power law was established for the dependence on intergalactic separation of the pair correlation function, with exponent $\gamma = 1.8$ [1]. Second, it was found that the data on higher order distributions supported the assumption that the nth order correlations are homogeneous functions of order $(N-1)\gamma$ [1].

In pioneering work, Bialin and Schaeffer [5] employed these properties to reconcile the assumption of homogeneity on large scales with fractality on intermediate and small scales. They demonstrated that the distribution of galaxies was not consistent with a normal fractal, but rather with a geometry that is characterized by two length scales, l_1 and l_2, where $l_o \gg l_1 \gg l_v$, and $l_v \gg l_2 \gg l_c$. Here the transition to homogeneity occurs at l_o, the maximum size of voids is l_v, and the typical size of clusters is l_c. They named this object "bifractal." More recently the approach has been successfully applied to the distribution of mass in halos [6].

In addition to the Hausdorff dimension, the existence of a hierarchy of generalized dimensions D_q which are closely related to the properties of correlation functions at small scales for $q \geqslant 2$, is now well established [7]. In particular, D_0 is simply the box counting dimension, D_1 is the information dimension, and D_2 is the correlation dimension. For a normal fractal with self-similar geometry at all scales, $D_{q+1} \leqslant D_q$ [7]. An unusual feature of bifractal geometry is the violation of this inequality, which can be taken as its signature [5,6].

A central question for astrophysics is whether it is possible to construct a single, consistent, dynamical model that (1) obeys physical law, (2) is homogeneous on large scales, (3) exhibits hierarchical clustering or aggregation, and (4) is characterized by bifractal geometry. While three-dimensional *N*-body simulations suggest this possibility [8], due to both algorithmic and numerical limitations at the present time their results must be considered inconclusive, and we will elaborate on this point later. As early as the 1930's, dynamical models of cosmic evolution were introduced that joined the Hubble flow with the gravitational field [1]. They were later rejected as possible sources of structure formation because, in the linear approximation, density fluctuations about a homogeneous background did not grow sufficiently rapidly to produce galaxies during the lifetime of the universe [1]. Here we revisit a version of one of these models with the goal of following the evolution of small initial fluctuations over the full nonlinear regime.

In the following we introduce a version of a de Sitter universe obeying classical dynamics appropriate to the postrecombination epoch. As in the relativistic Tolman-Bondi models [9], we simplify the geometry by assuming spherical symmetry about an observer. In addition, we examine a region sufficiently distant from the observer that the effects of curvature can be ignored. In contrast with most earlier treatments [1,9], we rescale both space and time to obtain a completely autonomous dynamical system. The model was first introduced by Rouet and Feix [10,11], who showed that aggregation was stimulated by excitations at the Jeans length, and computed a box counting dimension less than unity, suggesting fractal behavior. Here we use Vlasov-Poisson theory to characterize the central properties of the dynamics. We then use dynamical simulation to examine the consequences of a variety of initial conditions. We carry out a complete dimensional analysis of the phase plane and density distribution as time evolves, and show that the system exhibits bifractal geometry in all cases where the Jeans length is initially available to the fluctuations. We demonstrate that, for this model, the earliest time for aggregation is nearly independent of initial conditions or population. We use the value of this scaled time to estimate the earliest epoch for the appearance of galaxies in the universe, with surprising results.

II. CONSTRUCTION OF THE MODEL

Consider a spherically symmetric, homogenous, expanding universe with density $\rho(t)$ under conditions where Newtonian mechanics applies. Let $C(t/t_0)$ be the cosmological scale factor, so that the distance $l(t)$ between two objects at the time t is related to that at the earlier time t_0 by $l(t) = C(t/t_0)l(t_0)$. Here t_0 does not signify the big bang, but rather an arbitrary, later time where only gravitational phenomena play an important role in the cosmic evolution. To this expansion we must add a residual motion that is a small perturbation of the Hubble flow, but leads the system to a nonlinear regime.

From spherical symmetry, we only need to track a single coordinate, the radius. Thus our system elements are represented by concentric mass shells. The description can be further simplified by assuming that we are far from the center of symmetry and that the length of the system is small compared to the radius of the shells, so that we may replace them with planar sheets. Then the equation of motion of a sheet with coordinate x is simply

$$\frac{d^2x}{dt^2} = E(x,t), \tag{2.1}$$

where E is the gravitational field.

For the special case of an Einstein–de Sitter universe, there is a unique rescaling of space and time to a new frame in which the dynamical evolution is autonomous. Introduce new coordinates $\hat{x}$ and $\hat{t}$,

$$x = C(t)\hat{x}, \quad dt = A(t)^2 d\hat{t}, \tag{2.2}$$

where $C = (t/t_0)^{2/3}$ is the usual scale factor [1]. To insure that the transformed version of Eq. (2.1) is autonomous, we must then choose $A(t) = (t/t_0)^{1/2}$. The complete three-dimensional expansion is taken into account with this choice of $C(t)$. In the transformed frame the average density $\hat{\rho}$ is constant. Following standard practice, we choose the inverse Jeans frequency, defined by $\hat{\omega}_j^2 = 4\pi G\hat{\rho}$ as our unit of time (see [10] for a more complete discussion). The equation of motion for a mass sheet in the new frame then takes the form

$$\frac{d^2\hat{x}}{d\hat{t}^2} + \gamma\frac{d\hat{x}}{d\hat{t}} - \hat{x} = \hat{E}, \tag{2.3}$$

where $\hat{E}$ is the transformed field and the choice of a neutralizing background requires $\gamma = 1/\sqrt{2}$. Equation (2.3) describes the motion of a collisionless system of particles moving under their mutual gravitational field. From Gauss's law applied to uniform mass sheets, the field experienced by a particle on the line is simply proportional to the net difference in mass of the particles to its right and left. The transformations have induced both a linear friction and a constant, "negative mass" background density ρ_b. Thus the system is equivalent to a single component plasma with a drag force in which opposite charges repel and like charges attract.

In the mean field (Vlasov) limit, the system is amenable to a continuum description. Useful information can be had from the time dependant Vlasov equation, fixing the evolution of the density in the $x-v$ phase plane. For example, we easily find that the system energy decreases at a rate proportional to the kinetic energy, while the entropy decreases at the constant rate -2γ, and the Tsallis entropy decreases exponentially for $q>1$. This tells us that the mass is being concentrated in regions of decreasing area of the phase plane suggesting the development of structure. By asserting a Euclidean metric in the phase plane, we can also investigate local properties such as the directions of maximum stretching and compression, as well as the local vorticity. We find that the rate of separation between two nearby points is a maximum in the direction given by

$$\tan(2\theta) = (1+\rho+\rho_b)/\gamma, \tag{2.4}$$

where θ defines the local slope (angle made with the abscissa) in the phase plane. Thus, in regions of low density, we expect to see lines of mass being stretched with constant positive slope.

For a discrete population, the dynamics can be viewed as a sequence of particle crossings. Between an adjacent pair of crossings, Eq. (2.3) can be integrated analytically to yield an explicit solution for the position and velocity of each particle. Following the selection of an initial condition, an event driven algorithm was employed to compute the crossing sequence. The details of the algorithm are described elsewhere [10]. The evolution was followed until boundary effects became noticeable, typically in 15–20 dimensionless time units. In a few instances (see below) much longer simulations were carried out. The evolution of the system was systematically investigated with dynamical simulation for system populations of 10 000 and 50 000 particles (or sheets). Selected runs were also carried out with 500 000 sheets.

III. DYNAMICAL SIMULATION

Depending on which cosmology we select, the statistics of the dependence of velocity on position in the linear re-

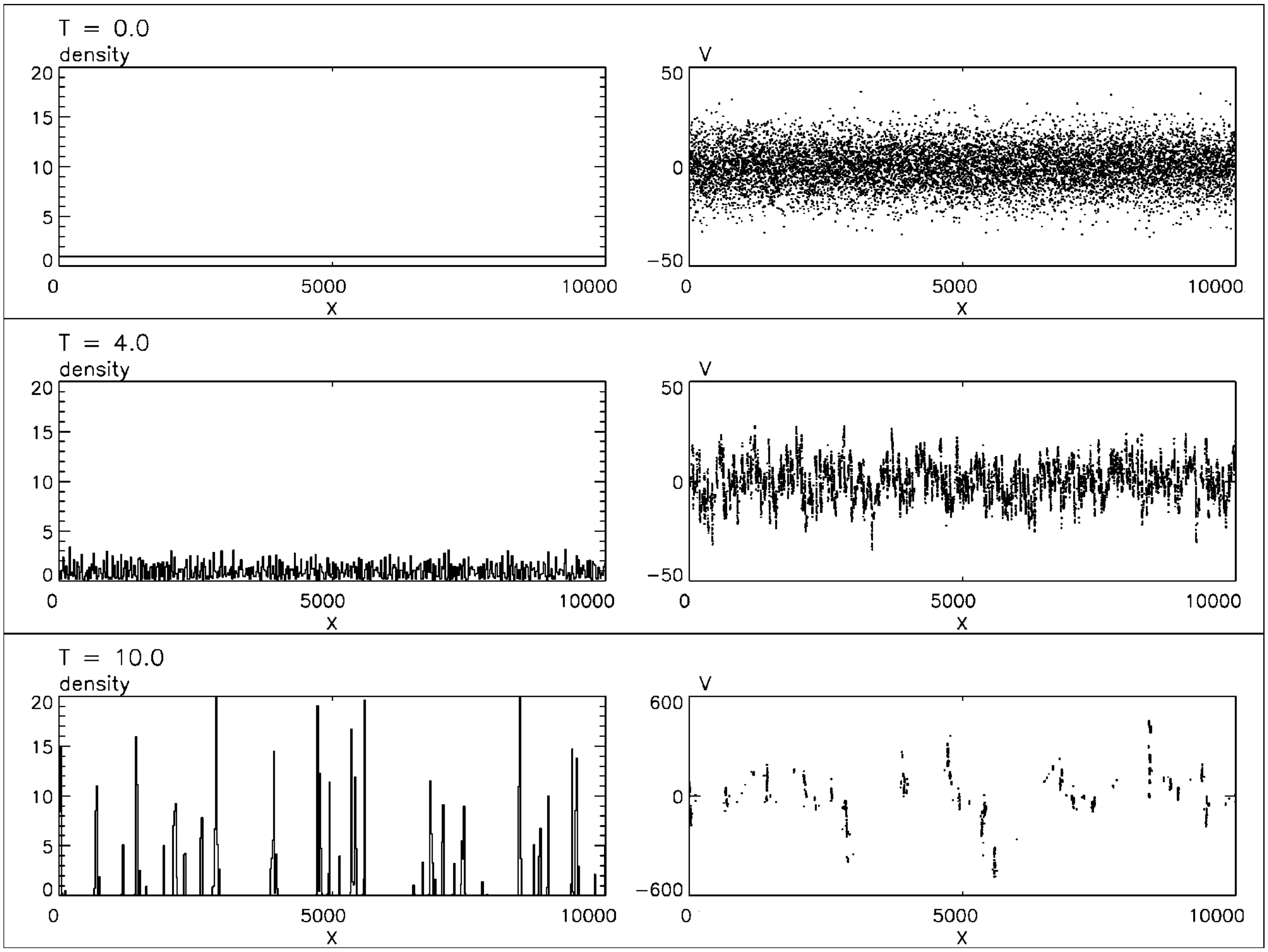

FIG. 1. Density and phase plane distribution for the initial condition, and after evolution for four and ten dimensionless time units, for $N=50\,000$ particles. The velocities are initially chosen at random from a Gaussian distribution with initial variance $L/\lambda_j=2000$.

gime is characterized as Gaussian, $1/f$ noise, or a Brownian motion of the normal or fractional variety [1]. In order to determine the robustness of the dynamics, the response of the system to several initial conditions was investigated. Here we discuss the two extremes: an initial Gausssian distribution in velocity, and a Brownian motion in position. In all cases, the particle positions were initially located equidistantly along the coordinate line. For the Gaussian (or isothermal) case, the velocity of each particle was independently selected from a normal distribution of mean zero and variance σ_0^2. The initial temperature was chosen such that the length of the system was about 2000 times the Jean's length, $\lambda_j=\sigma_0\omega_j$. In contrast, for the Brownian motion initial condition, the increment in velocity from one particle to another along the line is normally distributed. Thus in this case, the initial velocities of neighboring particles are strongly correlated.

As time evolved, visual inspection of the distribution of the cloud of points in the x-v phase plane, and their positions on the line, indicated they were similar for each initial condition, and we display the Gaussian (see Fig. 1). For short times, before crossings can occur, the field experienced by each particle is very weak, and we observe the exponential decrease in speed induced by the friction. However, as the number of crossings increases, the effects of instability become apparent. Typically, in about four dimensionless time units, two types of structure become obvious—lines and clumps. In the low density regions, the particles are distributed along a line of constant slope in the phase plane, as suggested by the stretching analysis discussed above, while in the high density regions they form clumps of roughly equal size. As the simulation goes forward in time, the process is repeated in hierarchical fashion, i.e., the clumps merge into bigger clumps. To test the role of Jean's length, we also prepared a much hotter system, where the Jean's length exceeded the system size. In that case, after the system cooled, a single clump formed near one boundary. However, even this distribution evidenced hierarchical layering around its center in the phase plane.

It is natural to assume that the apparently self-similar structure that develops in the phase plane and along the coordinate axis as time evolves has fractal geometry, but we will see that things aren't so simple. An earlier study of particle positions on the line found a box counting dimension of about 0.6 for an initial waterbag distribution (uniform on a rectangle in the phase plane) [11]. Since the structures that evolve are strongly inhomogeneous, to gain further insight we decided to carry out a multifractal analysis [7] in both the phase plane and the position coordinate. To accomplish this

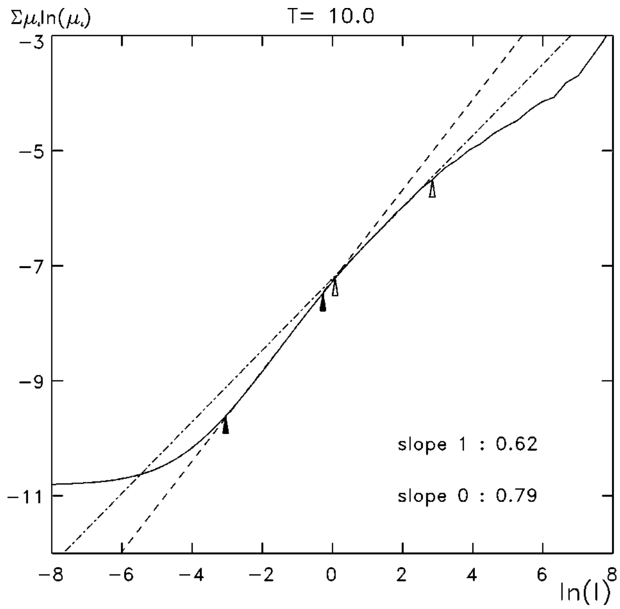

FIG. 2. Plot of $\Sigma \mu_i \ln(\mu_i)$ vs the log of the box size. The two dashed lines show the two regions for which the curve reveals a linear scaling regime. For smaller and larger scales (not shown) the slope takes on the obvious values of 0, and 1, respectively.

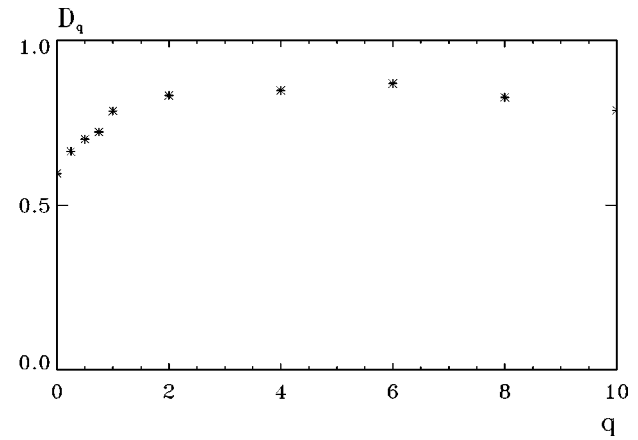

FIG. 3. Plot of D_q vs q for the same initial conditions and time as Fig. 2.

we partitioned each space into cells of length l. At each time of observation in the simulation, a measure $\mu_i = N_i(t)/N$ was assigned to cell i, where $N_i(t)$ is the population of cell i at time t and N is the total number of particles in the simulation. The generalized dimension of order q is defined by [7]

$$D_q = \frac{1}{q-1} \lim_{l\to 0} \frac{\ln C_q}{\ln l}, \quad C_q = \sum \mu_i^q. \tag{3.1}$$

As q increases above 0, the D_q provide information on the geometry of cells with higher population. If it exists, the scaling range of l is defined as the interval on which plots of $\ln C_q$ vs $\ln l$ are linear. Of course, for the special case of $q=1$, we plot $\Sigma \mu_i \ln \mu_i$ vs $\ln l$. If a scaling range can be found, D_q is obtained by taking the appropriate derivative. It is well established by proof and example that, for a normal, homogeneous, fractal, all of the generalized dimensions are equal, while for an inhomogeneous fractal, e.g., the Henon attractor, $D_{q+1} \leqslant D_q$ [7].

As expected, initially, and for a short time afterwards, all simulations showed a box counting dimension of two in the phase plane, and one along the coordinate axis. As time progressed, however, for each of the two initial conditions discussed above, at least one clear scaling range developed early in the simulation. For both the Gaussian and the Brownian motion, D_0 quickly converged on about 0.6 and remained there for most of the simulation. The size of the scaling range depended on both the elapsed time into the simulation and the value of q. We started our investigation by computing the first three generalized dimensions. We were surprised to observe that, in fact, $D_2 > D_0$ in all cases Moreover, for $q \geqslant 1$, a secondary weaker scaling range was also detected.

In Fig. 2 we plot C_1 vs $\ln l$ at the time $\hat{t}=10$ for the isothermal initial condition. We clearly see one dominant scaling range for small l, a second scaling range for intermediate l, and the possibility of a third range for larger l. It is suggestive that the transition between the first two scaling regimes occurs roughly at the Jeans length of the initial distribution. Since the size of the first clusters are approximately equal to the Jeans length, the suggestion is that the fractal geometry within the clusters differs from that of the less populous "voids." In Fig. 3 we plot D_q vs q for the same conditions and time as Fig. 2. We see that most of the change in dimension occurs when $0<q<1$. Although there is little change in D_q for $q>2$, the dominant scaling range grows progressively smaller with increasing q. This type of behavior was first inferred in a study of the observed correlations of galaxy positions by Bialin and Schaeffer [5] who named the geometry bifractal since it characterizes the superposition of two independent regular fractals. Subsequently it has also been observed in some three-dimensional N-body simulations [8].

IV. DISCUSSION AND CONCLUSIONS

In the last few years, the dynamics of a group of autonomous one-dimensional models has been studied for the purpose of gaining new insight concerning the development of hierarchical structures. These include the adhesion model [12], Burgers equation [13], and different versions of the system studied here, either with no scaling in position or time [14] (so there is neither a background nor friction), incomplete scaling in position [15](yielding a background but no friction), or with fractal initial conditions [16]. In the adhesion model, particles move on the line according to their mutual gravitational attraction. However, when they cross, they stick. In this system aggregation into a single large cluster occurs quickly, but a finite fraction of the system remains associated with smaller clumps for a long time [12]. Burgers equation has been carefully studied for a range of initial conditions that vary according to the correlation of initial velocities on the line [13]. For some initial states, shocks develop, yielding velocity portraits $v(x)$ similar to what we see in Fig. 1.

Very recently, structure formation was observed in the conservative one dimensional gravitational system, both with and without a uniform, ‘‘negative mass’’ (see above) background [14–16]. To get a better sense of how scaling influences the development of structure, we also performed simulations of these systems and examined their multifractal properties. The results were interesting: Similar, hierarchical, structures developed in each system. However, the generalized dimensions were larger in each case, about 0.8 for D_0, and bifractal behavior was much weaker than in the dissipative version studied here. In fact, with no background, we did not observe a bifractal structure and, with the background present, although we found $D_2 > D_0$, the difference was small. In each of these systems, the dimensions were less stable and varied with time. In the case without background, the fractal appearance washed out with the subsequent virialization. With the background present, the structure endured for a longer time. For contrast, we carried out a long simulation of the fully scaled system. We found that fractal structure and scaling endured long after the system retreated to a single cluster confined to a small region of the phase plane.

We observed earlier that, so long as the Jeans length was initially accessible to the system, the formation of structure occurred rapidly and robustly at about four dimensionless time units for all attempted initial conditions. Of course, this was in scaled time. Converting back to cosmic time, we simply find $t = t_o \exp(3/2)(\hat{t} - \hat{t}_0)$. If we take t_o as the time of recombination in a de Sitter universe, approximately 10^5–10^6 years [1], and scaled time $\hat{t} - \hat{t}_0 = 4.0$, we obtain a time in the range $(0.5–5) \times 10^9$ years for the appearance of the first galaxies. It may seem naive to use such a simple model to try to estimate the time when galaxies first appear. However, since (1) the origin of the model is three dimensional (see above), (2) the model consistently couples gravity with expansion, and (3) dynamical simulation shows that the results are similar under a large variety of initial conditions, we should not be overly surprised that what we find is of the correct order.

An interesting, and potentially useful, feature of the model is that it unambiguously exhibits what has been coined bifractal geometry. While this type of structure has been inferred from the study of correlation functions and ‘‘counts in cells’’ for the distribution of galaxies, the ability to construct the geometry with an autonomous dynamical system could yield additional insights. Although complete three-dimensional simulations could potentially yield more information, as a result of computational limitations it has proven difficult to obtain such unambiguous results from them [8,17]. This is not surprising if we consider that algorithms employed in three-dimensional simulations numerically cutoff both the short range singularity and infinite range of the Newtonian potential and, at the present time, typically employ 32–128 particles/dimension. In contrast, for the model considered here, it is not necessary to compromise the dynamics. Moreover, with our event driven algorithm we easily include 50 000 particles/dimension, or more. Since good statistics are essential for both determining the existence of scaling regimes and computing generalized dimensions with confidence, this feature is of critical importance.

In a larger work we will elucidate the multifractal features in more detail and study their connection with correlations in position and in the phase plane. Important questions for future work concern the number of possible scaling regimes, the existence of a scale on which homogeneity is established, and the connection with simulations in higher dimension.

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